Understanding Accessible ∞-categories

accessible ∞-categories $\Leftarrow$ 1-categories

idea: C = small ∞-category, κ = regular cardinal $\Longrightarrow$ $\mathrm{Ind}_\kappa(C) \cap \mathrm{Fun}(C^{op}, \mathcal{S}) = \mathrm{PSh}(C)$

equiv fu are of these:

$\mathcal{S}$ = ∞-groupoids

"κ-filtered ∞-category"

{1-cats $\Longrightarrow$ replace $\mathcal{S}$ with Set}

Example:

R=ring

$$\mathrm{Ind}_\omega(\mathrm{Mod}_R^{f.pres}) \simeq \mathrm{Mod}_R$$

ω = countable

$$\cup\!\mathsf{I}$$

$$\mathrm{Ind}_\omega(\mathrm{Mod}_R^{f.gen\ free}) \simeq \mathrm{Mod}_R^{flat}$$

(Adamek - Borceux - Lack - Rosicky)

$\leadsto$

class $\mathcal{U} \subseteq \mathrm{Cat}_\infty :=$ ∞ cat of small ∞-cats

$J \in \mathrm{Cat}_\infty$ · **$\mathcal{U}$-filtered** if

$$\mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \longrightarrow \mathcal{S}$$

preserves U-limits $\forall$ $\underline{U \in \mathcal{U}}$

ie, $U \xrightarrow{f} \mathrm{Fun}(J,\mathcal{S}) \Longleftrightarrow \hat{f} : J \times U \to \mathcal{S}$

$$\mathrm{colim}_J \lim_U \tilde{f} \xrightarrow[?]{} \lim_U \mathrm{colim}_J \tilde{f}$$

$J \in \mathrm{Cat}_\infty$ is **weakly $\mathcal{U}$-filtered** if

$\forall$ $f : U^{op} \to J$, $\forall$ $U \in \mathcal{U}$, the slice

$J_{f/}$ is weakly contractible as a s.set

obj:

$$\begin{tikzcd}
U^{op} \arrow[r, "f"] \arrow[d, hook] & J \\
(U^{op})^{\triangleright} \arrow[ur, dashed] &
\end{tikzcd}$$

right cone

Straightening/Unstraightening:

$$\begin{matrix} D \\ p\downarrow \\ C \end{matrix} \text{ left fibration } \iff \begin{tikzcd} D \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & S_* \arrow[d, "P_{univ}"] \\ C \arrow[r, "f"'] & S \end{tikzcd}$$

functor

$$\lim_C f = \{\text{sections of } p\} \in S$$

$$\boxed{\mathrm{colim}_C f \underset{w.e.}{\simeq} D}$$

$$f : U^{op} \to J \implies \begin{tikzcd} J_{f/} \arrow[r] \arrow[d, "\text{left fib}"'] & S_* \arrow[d] \\ J \arrow[r, "g"'] & S \end{tikzcd}$$

$$\implies \text{classifies } J \to S$$

$u \in U$

$$g := \lim_{u \in U} \mathrm{Map}_J(f(u), -)$$ ✓ limit of corepresentable functors

$$J_{f/} \underset{}{\overset{w.e.}{\simeq}} \mathrm{colim}_J \lim_U \mathrm{Map}(f, -)$$

$$\downarrow$$

$$* \underset{w.e.}{\simeq} \lim_U \underline{\mathrm{colim}_J \mathrm{Map}(f, -)}$$

$$\Rightarrow \quad \mathrm{Filt}_U \subseteq w\mathrm{Filt}_U \subseteq \mathrm{Cat}_\infty$$

$$\uparrow ?? \qquad \pi$$

$\mathcal{U} \subseteq \mathrm{Cat}_\infty$ is **sound** if

$$\mathrm{Filt}_{\mathcal{U}} = w\mathrm{Filt}_{\mathcal{U}}$$

$\mathcal{U} \subseteq \mathrm{Cat}_\infty$ is a **doctrine** if

$\exists$ a set S of ∞-cats st every $U \in \mathcal{U}$ is equiv to some $U' \in S$

Examples:

- $\mathcal{U} = \underline{\mathrm{Cat}_\infty} \rightsquigarrow \mathrm{Filt}_{\mathcal{U}} = \left\{ J \;\middle|\; \begin{array}{l} \mathrm{colim}_J : \mathrm{Fun}(J, \mathcal{S}) \to \mathcal{S} \\ \text{preserves all limits} \end{array} \right\}$

  $= \{ J \mid J^{\mathrm{idem}} \text{ has a term obj} \}$

  $= \mathrm{wFilt}_{\mathrm{Cat}_\infty}$

- $\mathcal{U} = \emptyset \rightsquigarrow \mathrm{Filt}_\emptyset = \mathrm{Cat}_\infty = \mathrm{wFilt}_\emptyset$

- $\mathrm{term} = \{\emptyset\} \sim \underline{\mathrm{Filt}_{\mathrm{term}}} = \{ J \mid \mathrm{colim}_J * \simeq * \}$

  i.e. $\overset{\text{contractible } \infty\text{-cats}}{J} \;\; \overset{\shortparallel}{\underline{\mathrm{wFilt}_{\mathrm{term}}}}$

- $\mathrm{fin}X := \{\text{finite sets}\} \sim \mathrm{Filt}_{\mathrm{fin}X} = \text{sifted } \infty\text{-categories}$

  $= \mathrm{wFilt}_{\mathrm{fin}X}$

- $\mathrm{pb} := \{ \bullet \to \overset{\displaystyle\bullet}{\underset{}{\downarrow}}\!\!\bullet \} \sim \mathrm{Filt}_{\mathrm{pb}} = \mathrm{wFilt}_{\mathrm{pb}}$

  "distilled"

  [$\underline{1\text{-cat}}$: $\mathrm{Filt}_{\mathrm{pb}}$ not sound]

$\kappa\text{-small} := \{ U \in \mathrm{Cat}_\infty ,\ \exists\ K \xrightarrow[\text{cat eq}]{\simeq} U ,\ K\ \kappa\text{-small sSet} \}$

κ = regular cardinal

$$\Rightarrow \quad \mathrm{Filt}_{\kappa\text{-small}} = \mathrm{wFilt}_{\kappa\text{-small}} = \mathrm{Filt}_\kappa .$$

say J is κ-filtered if

$\forall\ f : K \to J$ $\quad J_{f/} \neq \emptyset$

K κ-small,

κ is regular if infinite and if $\{S_i\}_{i \in I}$

$|S_i| < \kappa ,\ |I| < \kappa \implies |\bigcup_i S_i| < \kappa$

κ infinite card.

$$\mathrm{Filt}_{\kappa\text{-sm}} \overset{?}{\subseteq} \mathrm{wFilt}_{\kappa\text{-sm}} \overset{?}{\subseteq} \mathrm{Filt}_\kappa$$

if κ not regular $\rightsquigarrow$ κ^+ is regular

and $\quad \mathrm{Filt}_{\kappa\text{-small}} = \mathrm{Filt}_{\kappa^+\text{-small}}$ ✓

Unsound doctrine:

$$eq := \{ \underline{\bullet \rightrightarrows \bullet} \}$$

$$\mathrm{Filt}_{eq} \subsetneq \mathrm{wFilt}_{eq}$$

$$\not\ni \qquad\qquad \ni$$

$$\left(\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right)_J \qquad\qquad J_{f/}$$

$$\mathrm{Filt}_{eq} \simeq \;?? \;\supseteq\; \mathrm{Filt}_{\mathrm{conn},\, \omega\text{-small}}$$

$$\| $$

$$\{ \text{coproducts of } \omega\text{-filt } \infty\text{-cats} \}$$

$$
\begin{tikzcd}
\mathcal{U} \arrow[rr, squiggly] \arrow[d, phantom, sloped, "\subseteq"] & & \mathrm{Filt}_{\mathcal{U}} \arrow[lld, squiggly] \arrow[d, equal] \\
\overline{\mathcal{U}} \arrow[rr, squiggly, dash] \arrow[d, phantom, sloped, "\subseteq"] & & \mathrm{Filt}_{\overline{\mathcal{U}}} \arrow[lld, squiggly] \\
\overline{\overline{\mathcal{U}}} & &
\end{tikzcd}
$$

$$\overline{\mathcal{U}} = \{ U \mid \operatorname{colim}_J \lim_U = \lim_U \operatorname{colim}_J \ \forall \ J \in \mathrm{Filt}_{\mathcal{U}} \}$$

a **regular class** is one of form $\overline{\mathcal{U}}$ where $\mathcal{U}$ is a doctrine

$\updownarrow$

a **coregular class** is one of form $\mathrm{Filt}_{\overline{\mathcal{U}}}$

κ inf card $\implies$ $\overline{\kappa\text{-small}} \,/\, \mathrm{Filt}_{\overline{\kappa\text{-small}}}$

regular cardinals $\subseteq$ regular classes

$$\underline{\mathrm{Filt}_{\mathcal{U}} \cap \mathcal{S}} \simeq \mathrm{Null}_{B\mathcal{U}}$$ ← "nullity classes"

$$\mathrm{Null}_{B\mathcal{U}} = \{ X \in \mathcal{S} \mid \mathrm{Map}(BU, X) \xleftarrow{\simeq} X \quad \forall\, U \in \mathcal{U} \}$$

$U \to BU$: groupoid completion

$$\begin{array}{cc} \cap & \cap \\ \mathrm{Cat}_\infty & \mathcal{S} \end{array}$$

$$\mathrm{Null}_{\{S^{n+1}\}} = \{ n\text{-truncated spaces} \}$$

$\mathcal{U}$ doctrine $\Longrightarrow$ full subcats of $PSh(C) = Fun(C^{op}, S)$

if C^{op} has $\mathcal{U}$-limits

$$PSh(C)$$
$$\cup\!\!|$$
$$wInd_{\mathcal{U}}(C) := \{ X \mid C/X \in wFilt_{\mathcal{U}} \} \quad ??$$
$$\cup\!\!|$$
$$Lim_{\mathcal{U}}(C) := \{ X \mid X : C^{op} \to S \text{ preserves } \mathcal{U}\text{-limits} \} \quad \checkmark$$

$\Rightarrow$ (C/X) has $\mathcal{U}^{op}$-colimits

$$\cup\!\!|$$
$$Flat_{\mathcal{U}}(C) := \{ X \mid \hat{X} : Fun(C, S) \to S \text{ preserves all } \mathcal{U}\text{-limits} \} \quad \checkmark$$
$$\cup\!\!|$$
$$Ind_{\mathcal{U}}(C) := \{ X \mid C/X \in Filt_{\mathcal{U}} \} \quad \Longleftarrow$$
$$\cup\!\!|$$
$$C$$

all are iff $\mathcal{U}$ is sound

$$C/X := C \times_{PSh(C)} PSh(C)_{/X} \qquad \text{"comma cat"}$$

$$\begin{array}{c} C/X \\ \pi \big\downarrow \ \text{right fibration} \\ C \end{array} \quad \Longleftrightarrow \quad C^{op} \xrightarrow{X} S$$

$$\hat{X} := LKan_{\rho} X \qquad \rho : C^{op} \to Fun(C, S)$$

$$\begin{tikzcd}
 & S & \\
C^{op} \arrow[ur, "X"] \arrow[r, "\rho"'] \arrow[r, phantom, bend left=20, "\circlearrowleft"] & Fun(C, S) \arrow[u, "\hat{X} := LKan_{\rho} X"'] \arrow[r, "\pi^{*}"'] \arrow[r, phantom, bend left=30, "\simeq"] & Fun(C/X, S) \arrow[ul, bend right, "colim_{C/X}"']
\end{tikzcd}$$

Prop: $\mathrm{Ind}_{\mathcal{U}}\overset{X}{\hat{}}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$

is stable under $\mathcal{U}$-filtered colimits

$$\underline{X} = \mathrm{colim}_{\mathcal{C}/X} \left[\rho \circ \pi : \mathcal{C}/X \xrightarrow{\pi} \mathcal{C} \xrightarrow{\rho} \mathrm{PSh}(\underline{\mathcal{C}})\right]$$

Cor. $\mathrm{Ind}_{\mathcal{U}}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ is smallest full. which contains $\mathcal{C}$ and is stable under $\mathcal{U}$-filt colims

Cor: $\mathcal{C} \overset{\rho}{\hookrightarrow} \mathrm{Ind}_{\mathcal{U}}(\mathcal{C})$ is the "free $\mathcal{U}$-filtered colimit completion" of $\mathcal{C}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, hook, "\rho"'] & \mathcal{A} \\
\underline{\mathrm{Ind}_{\mathcal{U}}(\mathcal{C})} \arrow[ur, dashed, "\exists!\ ?"']
\end{tikzcd}$$

$\mathcal{A}$ ⇐ ∞-cat with all $\mathcal{U}$-filt colimits

$\tilde{f}$ = pres $\mathcal{U}$-filt colimits

$\mathcal{A}$ = has all $\mathcal{U}$-filt colims

say $A \in \mathcal{A}$ is $\mathcal{U}$-compact if

$\mathrm{Map}_{\mathcal{A}}(A, -) : \mathcal{A} \longrightarrow \mathcal{S}$ commutes with $\mathcal{U}$-filt colims

<u>Prop:</u>

$$\begin{tikzcd}
C \arrow[r, hook, "f", "\text{full. faith}"'] \arrow[d, "p"'] & \mathcal{A}^{\mathcal{U}\text{cpt}} \subseteq \mathcal{A} \\
\mathrm{Ind}_{\mathcal{U}}(C) \arrow[ur, dashed, "?" description, "\tilde{f}"'] &
\end{tikzcd}$$

$\leftarrow$ is also fully faithful

<u>Fact</u>: $\left.\begin{array}{l} \mathcal{U}\text{-doctrine} \\ C \in \mathrm{Cat}_\infty \end{array}\right\} \Longrightarrow \mathrm{Flat}_{\mathcal{U}}(C)$ is accessible.

$(\simeq \mathrm{Ind}_\kappa(D))$

$\mathcal{A}$ is accessible iff $\simeq \mathrm{Flat}_{\mathcal{U}}(C)$ for some $\mathcal{U}, C$

Is $\mathrm{Ind}_{\mathcal{U}}(C)$ accessible?

YES $\Longrightarrow$ another char of acc ∞-cats

NO $\Longrightarrow$ strict generalization of acc ∞-cats

$\mathrm{Filt}_{eq} \quad =: \quad \mathrm{Ind}_{eq}(C)$ acc??

$\mathrm{Filt}_{\mathcal{U}} \supseteq \mathrm{Filt}_\kappa$ for some κ.

<u>idea</u>: (if YES): $\mathcal{U}$ doctrine, find $\mathcal{U} \subseteq \mathcal{V}$ doctrine

st $\overline{\mathcal{U}} = \overline{\mathcal{V}}$

so that $\mathcal{V}$ is sound

$\mathrm{Filt}_{\mathcal{U}} = \mathrm{Filt}_{\mathcal{V}}$ $\qquad$ $w\mathrm{Filt}_{\mathcal{U}} \supseteq w\mathrm{Filt}_{\mathcal{V}}$

$$\mathrm{Filt}_{U \cap S} = \mathrm{Null}_{BU}$$

$F : J \times U \to S$ $\rightsquigarrow$ $*: J \times U \to S$

$$\begin{tikzcd}
\operatorname{colim}_J \lim_U F \arrow[r, "\cdot"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim_U \operatorname{colim}_J F \arrow[d] \\
BJ \arrow[r, "\cdot"'] & \mathrm{Map}(BU, BJ)
\end{tikzcd}$$

F $\leftarrow$ $*$

if $J = BJ$, then this a p.b. square

$J = \circ$ and $U = (\cdot \quad \cdot)$

$$\begin{matrix} F_1 & F_2 \\ \downarrow & \downarrow \\ BJ & BJ \end{matrix}$$

$$\begin{tikzcd}
F_1 \times_{BJ} F_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F_1 \times F_2 \arrow[d] \\
BJ \arrow[r] & BJ \times BJ
\end{tikzcd}$$