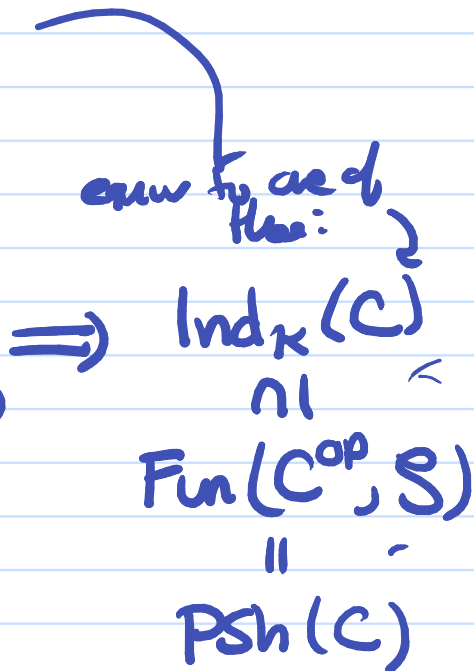


# Understanding Accessible $\infty$ -categories

accessible  $\infty$ -categories  $\Leftarrow$   
1-categories

idea:  $C =$  small  $\infty$ -category.  
 $\kappa =$  regular cardinal



$S = \infty$ -groupoids

" $\kappa$ -filtered  $\infty$ -category"

[1-cats  $\Rightarrow$  replace  $S$  with set]

Example:

$R = \text{ring}$

$$\text{Ind}_w(\text{Mod}_R^{\text{f.pres}}) \simeq \text{Mod}_R$$

$w =$  countable

UI

$$\text{Ind}_w(\text{Mod}_R^{\text{f.g. free}}) \simeq \text{Mod}_R^{\text{flat}}$$

( Adamek - Borceux - Lack - Rosicky )

→

class  $\mathcal{U} \subseteq \text{Cat}_\infty :=$   $\infty$  cat of small  $\infty$ -cats

$J \in \text{Cat}_\infty$  :  **$\mathcal{U}$ -filtered.** if

$\text{colim}_J : \text{Fun}(J, \mathcal{S}) \rightarrow \mathcal{S}$

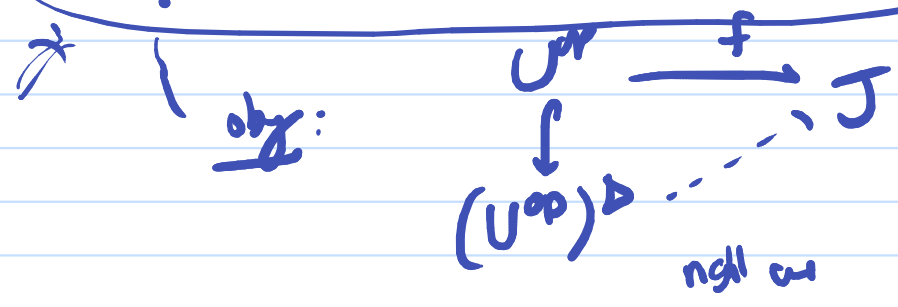
preserves  $\mathcal{U}$ -limits  $\forall \underline{U} \in \mathcal{U}$

ie,  $\mathcal{U} \xrightarrow{f} \text{Fun}(J, \mathcal{S}) \Leftrightarrow \hat{f} : J \times \mathcal{U} \rightarrow \mathcal{S}$   
 $\text{colim}_J \lim_{\mathcal{U}} \hat{f} \xrightarrow{??} \lim_{\mathcal{U}} \text{colim}_J \hat{f}$

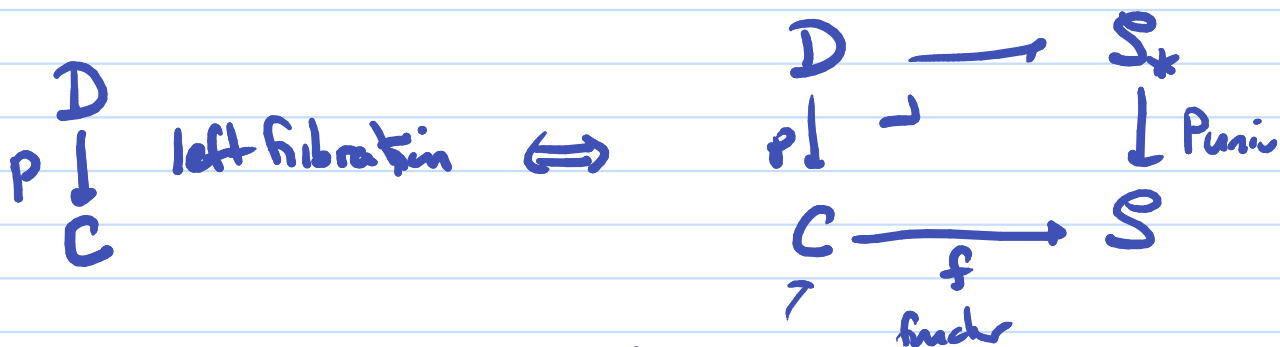
$J \in \text{Cat}_\infty$  is **weakly  $\mathcal{U}$ -filtered.** if

$\forall f : U^{\text{op}} \rightarrow J, \forall U \in \mathcal{U}$ , the slice

$J_{f/}$  is weakly contractible as a s.set



# Straightening/Unstraightening:



$$\lim_C S = \{ \text{sections of } p \} \in S$$

$$\text{colim}_C f \cong D \text{ w.e.}$$

$$f: U^{\text{op}} \rightarrow \mathcal{J} \implies \begin{array}{ccc} \mathcal{J}_{f/} & \xrightarrow{\quad} & S_* \\ \downarrow \text{left fib} & & \downarrow \\ \mathcal{J} & \xrightarrow{g} & S \end{array}$$

$$\implies \text{classifying } \mathcal{J} \rightarrow S$$

$$g := \lim_{u \in U} \text{Map}_{\mathcal{J}}(f(u), -) \quad \checkmark \text{ limit of corepresentable functors}$$

$$\mathcal{J}_{f/} \stackrel{\text{w.e.}}{=} \text{colim}_{\mathcal{J}} \lim_U \text{Map}(f, -)$$

$$\downarrow$$

$$* \stackrel{\text{w.e.}}{=} \lim_U \text{colim}_{\mathcal{J}} \text{Map}(f, -)$$

$$\implies \text{Filt}_U \subseteq \text{wFilt}_U \subseteq \text{Cartoo}$$

$$\uparrow \quad \uparrow$$

$$?? \quad \uparrow$$

$\mathcal{U} \subseteq \text{Cats}$  is **sound** if

$$\text{Filt}_{\mathcal{U}} = \omega\text{Filt}_{\mathcal{U}}$$

$\mathcal{U} \subseteq \text{Cats}$  is a **doctrine** if

$\exists$  a set  $S$  of  $\omega$ -cats st every  $U \in \mathcal{U}$   
is equiv to some  $U' \in S$



$\kappa$ -small :=  $\{ U \subseteq \text{Cats}, \exists K \xrightarrow[\text{cat eq}]{} U, K \text{ } \kappa\text{-small sSet} \}$

$\kappa$ -regular cardinal

$$\Rightarrow \text{Filt}_{\kappa\text{-small}} = \text{wFilt}_{\kappa\text{-small}} = \text{Filt}_{\kappa}$$

say  $J$  is  $\kappa$ -filtered if

$$\forall f: K \rightarrow J \\ K \text{ } \kappa\text{-small,}$$

$$J_{S_i} \neq \emptyset$$

$\kappa$  is **regular** if infinite and if  $\{S_i\}_{i \in I}$

$$|S_i| < \kappa, |I| < \kappa \Rightarrow \left| \bigcup_i S_i \right| < \kappa$$

$\kappa$  infinite card.

$$\text{Filt}_{\kappa\text{-sm}} \stackrel{?}{\subseteq} \text{wFilt}_{\kappa\text{-sm}} \stackrel{?}{\subseteq} \text{Filt}_{\kappa}$$

if  $\kappa$  not regular  $\sim \kappa^+$  is regular

$$\text{and } \text{Filt}_{\kappa\text{-small}} = \text{Filt}_{\kappa^+\text{-small}}$$

# Unsound doctrine:

$$eq := \{ \cdot \Rightarrow \cdot \}$$

$$Filt_{eq} \not\subseteq wFilt_{eq}$$

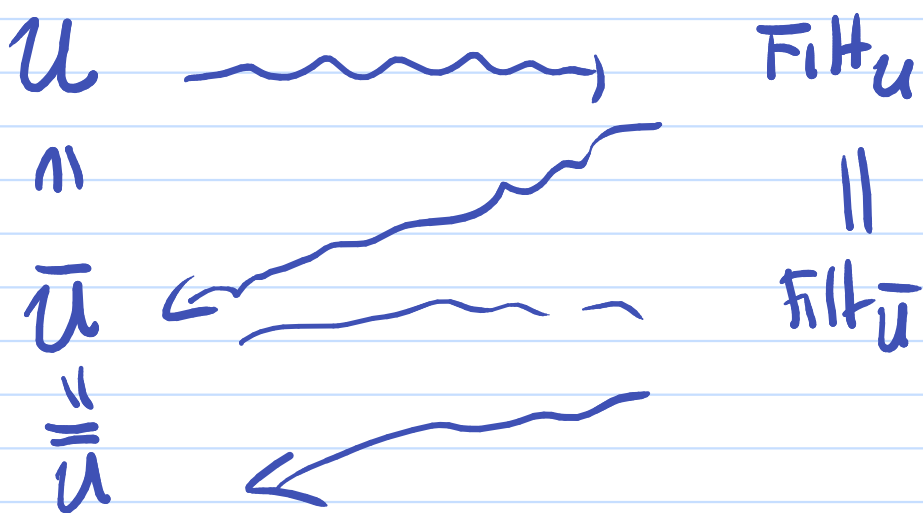
~~$\subseteq$~~

$\hookrightarrow$

$J_{\neq}$

$$\left( \begin{array}{c} \cdot \\ \downarrow \\ \cdot \end{array} \rightarrow \cdot \right)$$

$$Filt_{eq} \simeq ?? \supseteq \begin{array}{l} Filt_{\text{conn, w-small}} \\ \parallel \\ \{ \text{coproducts of w-filt} \\ \text{as-cats} \} \end{array}$$



$$\overline{\mathcal{U}} = \{ \mathcal{V} \mid \text{colim}_J \text{lim}_\mathcal{U} = \text{lim}_\mathcal{U} \text{colim}_J \ \forall J \in \text{Filt } \mathcal{U} \}$$

a **regular class** is one of form  $\overline{\mathcal{U}}$   
 where  $\mathcal{U}$  is a doctrine

a **coregular class** is one of form  $\text{Filt } \overline{\mathcal{U}}$

$$\kappa \text{ inf card} \implies \overline{\kappa\text{-small}} / \text{Filt}_{\overline{\kappa\text{-small}}}$$

$$\text{regular cardinals} \subseteq \text{regular classes}$$



$$\underline{\text{Filt}_U \cap S} \cong \underset{\text{ii}}{\text{Null}}_{BU} \quad \checkmark \quad \text{"nullity class"}$$

$$\{X \in S \mid \text{Map}(BU, X) \xrightarrow{\cong} X \quad \forall U \in \mathcal{U}\}$$

$$U \rightarrow BU \text{ : groupoid completion}$$

$$\hat{\text{Cat}}_U \quad \hat{S}$$

$$\text{Null}_{S \text{ null}} = \left( \begin{array}{l} n\text{-truncated} \\ \text{spaces} \end{array} \right)$$

$U$  doctrine  $\implies$  full subcat of  $\text{Psh}(C) = \text{Fun}(C^{\text{op}}, S)$   
 if  $C^{\text{op}}$  has  $U$ -limits

$\text{Psh}(C)$

$U$

$\text{wFilt}_U(C) := \{X \mid C/X \in \text{wFilt } U\}$  ??

$\text{Lim}_U(C) := \{X \mid X: C^{\text{op}} \rightarrow S \text{ preserves } U\text{-limits}\}$   $\implies (C/X) \text{ has } U\text{-limits}$  ✓

$\text{Flat}_U(C) := \{X \mid \hat{X}: \text{Fun}(C, S) \rightarrow S \text{ preserves all } U\text{-limits}\}$  ✓

$\text{Ind}_U(C) := \{X \mid C/X \in \text{Filt } U\}$   $\leftarrow$

$U$   
 $C$

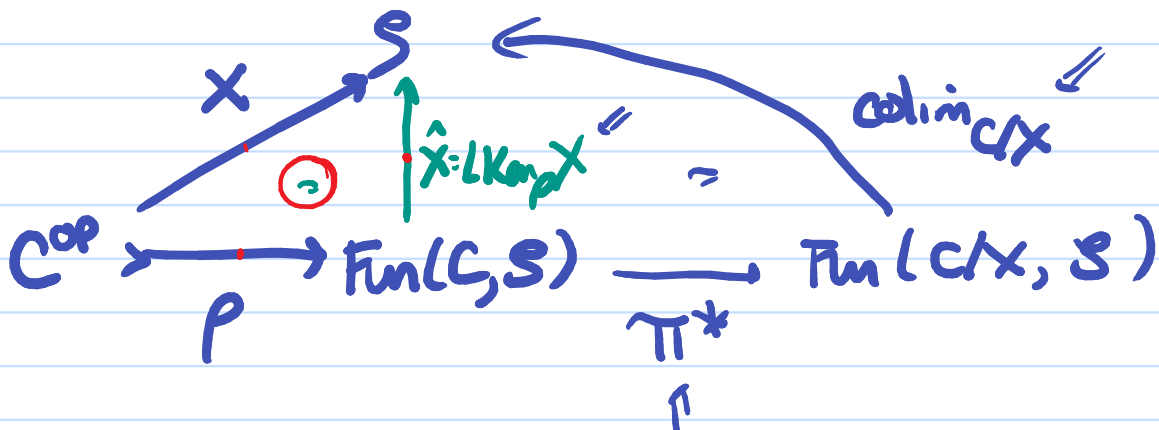
$C/X := C \times_{\text{Psh}(C)} \text{Psh}(C)_{/X}$  "comma cat"

$\pi \downarrow$  right fibration  
 $C$

$C^{\text{op}} \xrightarrow{X} S$

$\hat{X} := \text{LKan}_p X$

$p: C^{\text{op}} \rightarrow \text{Fun}(C, S)$

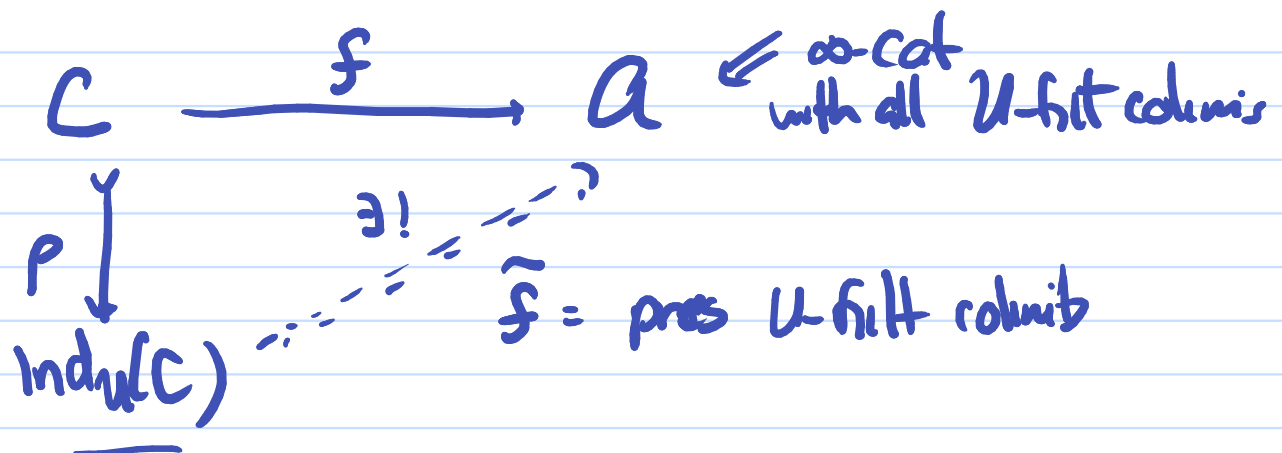


Prop:  $\text{Ind}_u^x(C) \in \text{PSh}(C)$   
 is stable under  $U$ -filtered colimits

$$X = \text{colim}_{C/X} (P \circ \pi: C/X \xrightarrow{\pi} C \xrightarrow{P} \text{PSh}(C))$$

Cor.  $\text{Ind}_u(C) \in \text{PSh}(C)$  is smallest full.  
 which contains  $C$  and is stable under  
 $U$ -filt colimits

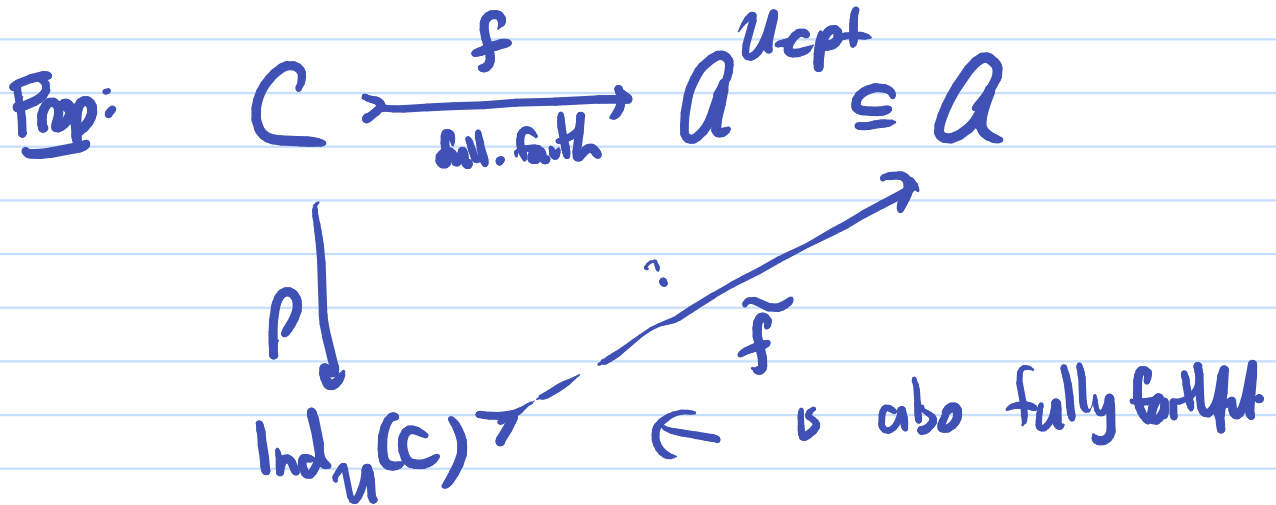
Cor:  $C \xrightarrow{P} \text{Ind}_u(C)$  is  
 the "free  $U$ -filtered colimit completion" of  $C$



$\mathcal{A} =$  has all  $\mathcal{U}$ -filt colims

say  $A \in \mathcal{A}$  is  $\mathcal{U}$ -compact if

$\text{Map}_{\mathcal{A}}(A, -) : \mathcal{A} \rightarrow \mathcal{S}$  commutes  
with  $\mathcal{U}$ -filt colims



Fact:  $\left. \begin{array}{l} U\text{-doctre} \\ C \in \text{Cats} \end{array} \right\} \Rightarrow \text{Flat}_U(C) \text{ is accessible.}$   
 $(\cong \text{Ind}_K(D))$

$\mathcal{A}$  is accessible iff  $\cong \text{Flat}_U(C)$  for some  $U, C$

Is  $\text{Ind}_U(C)$  accessible?

YES

another char of  
acc co-cats

NO

strict generalization  
of acc co-cats

↪

$\text{Filt}_{\text{eq}} = \text{Ind}_{\text{eq}}(C) \text{ acc??}$

$\text{Filt}_U \cong \text{Filt}_V$  for some  $\mathcal{X}$

idea: (if YES):  $U$  doctre, find  $U \subseteq V$  doctre

st  $\bar{U} = \bar{V}$

so that  $V$  is sound

$\text{Filt}_U = \text{Filt}_V$

$w\text{Filt}_U \cong w\text{Filt}_V$

$$\text{Filt}_u \cap S = \text{Null}_{BU}$$

$$F: J \times U \rightarrow S$$

$$\Rightarrow * : J \times U \rightarrow S$$

$$\text{colim}_J \lim_u F \xrightarrow{\quad} \lim_u \text{colim}_J F \quad F$$

$$\begin{array}{ccc} \downarrow & \downarrow & \leftarrow \\ BJ & \xrightarrow{\quad} & \text{Map}(BU, BJ) \quad \downarrow \end{array}$$

if  $J = BJ$ , then this has a p.b. square

$$J = \text{and} \quad U = (\cdot \cdot) \quad \begin{array}{cc} F_1 & F_2 \\ \downarrow & \downarrow \\ BJ & BJ \end{array}$$

$$\begin{array}{ccc} F_1 \times F_2 & \xrightarrow{\quad} & F_1 \times F_2 \\ \downarrow & \downarrow & \\ BJ & \xrightarrow{\quad} & BJ \times BJ \end{array}$$